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SOME OPEN MAPPING THEOREMS FOR MARGINALS

BY

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ABSTRACT. Let S and T be compact Hausdorff spaces and let $P(S)$, $P(T)$ and $P(S \times T)$ denote the collection of probability measures on S, T and $S \times T$, respectively. Given a probability measure μ on $S \times T$, set $\pi\mu = (\alpha, \beta)$ where α and β are the marginals of μ on S and T. We prove that the mapping $\pi: P(S \times T) \to P(S) \times P(T)$ is norm open and weak* open. An analogous result for $L_1(X \times Y, \mu \times \nu)$ where (X, μ) and (Y, ν) are probability spaces is established.

1. **Introduction.** Let S be a compact Hausdorff space. Let $C(S)$ denote the algebra of continuous real valued functions on S and let $M(S)$ denote the linear space of real valued regular Borel measures on S of finite total variation. We identify $M(S)$ with the dual of $C(S)$. Let $P(S)$ denote the probability measures on S. In [1], Ditor and the author obtained open mapping theorems for a naturally induced mapping between spaces of probability measures on compact sets. Namely, let S and T be compact Hausdorff spaces and let $\phi: S \to T$ be continuous and onto. Then ϕ induces a mapping $\pi: P(S) \to P(T)$ defined by $\pi\mu(V) = \mu(\phi^{-1}(V))$ for each Borel subset V of T. The following results are established in [1].

(*) $\quad \pi: P(S) \to P(T)$ is norm open.

(**) $\quad \pi: P(S) \to P(T)$ is weak* open iff $\phi: S \to T$ is open.

One in fact obtains the following quantitative result. Let $\mu \in P(S)$ and set $\alpha = \pi\mu$. Given $\beta \in P(T)$, there exists $\nu \in P(S)$ such that $\beta = \pi\nu$ and $\|\mu - \nu\| = \|\alpha - \beta\|$.

In this paper, we consider the open mapping properties of a naturally induced operator of interest in probability theory [3], [5]. Let S and T be compact Hausdorff spaces. Define $\pi: M(S \times T) \to M(S) \times M(T)$ by $\pi\mu = (\alpha, \beta)$ where α and β are the marginals of μ on S and T, i.e., $\alpha(U) = \mu(U \times T)$ and $\beta(V) = \mu(S \times V)$ for each Borel subset U of S and V of T. The collection of all marginals is

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$$M(S,\ T) = \{(\alpha,\ \beta) \in M(S) \times M(T)\colon \alpha(S) = \beta(T)\}.$$

In §2, we establish our main results.

(1) Let $\mu \in P(S \times T)$ and set $\pi\mu = (\alpha, \beta)$. Given marginals (λ, ρ) in $P(S) \times P(T)$, there exists $\nu \in P(S \times T)$ such that $\pi\nu = (\lambda, \rho)$ and $\|\mu - \nu\| \le \|\alpha - \lambda\| + \|\beta - \rho\|$. Hence, $\pi\colon P(S \times T) \to P(S) \times P(T)$ is norm open.

(2) $\pi\colon P(S \times T) \to P(S) \times P(T)$ is weak* open.

In §3, we consider the analogous case of $L_1(X \times Y,\ \mu \times \nu)$ where (X, μ) and (Y, ν) are probability spaces. We also establish an open mapping result for the conditional expectation operator.

2. Probability measures on $S \times T$.

In order to prove our main results (1) and (2), we first establish a quantitative version for the case when S and T are finite. An alternate proof of Lemma 2.1 is provided by the argument in 3.2. The type of combinatorial argument given below is needed to extend the result to finite products [6].

Lemma 2.1. *Suppose S and T are finite sets. Let $\mu \in P(S \times T)$ and set $\pi\mu = (\alpha, \beta)$. Fix marginals (λ, ρ) in $P(S) \times P(T)$. Then there exists $\nu \in P(S \times T)$ satisfying $\pi\nu = (\lambda, \rho)$ and $\|\mu - \nu\| \le \|\alpha - \lambda\| + \|\beta - \rho\|$.*

Proof. First consider the case where $\rho = \beta$ and where λ and α differ at exactly two points say s_1 and s_2. We may assume $\alpha(s_1) > \lambda(s_1)$ and so $\alpha(s_2) < \lambda(s_2)$. Set $\epsilon = \alpha(s_1) - \lambda(s_1)$ and $\delta = \alpha(s_1)$. Now define $\nu(s, t) = \mu(s, t)$ if $s \ne s_1$ and $s \ne s_2$ and define $\nu(s_1, t) = (\delta - \epsilon)/\delta \cdot \mu(s_1, t)$ and $\nu(s_2, t) = \mu(s_2, t) + (\epsilon/\delta) \cdot \mu(s_1, t)$. Then $\pi\nu = (\lambda, \rho)$ and $\|\mu - \nu\| = 2\epsilon = \|\alpha - \lambda\|$.

Now fix marginals (λ, ρ) in $P(S) \times P(T)$. Suppose $\alpha \ne \lambda$. Choose s_1, s_2 in S with $\alpha(s_1) > \lambda(s_1)$ and $\alpha(s_2) < \lambda(s_2)$. Set

$$\epsilon = \min\{\alpha(s_1) - \lambda(s_1),\ \lambda(s_2) - \alpha(s_2)\}.$$

Now define α' by $\alpha'(s) = \alpha(s)$ if $s \ne s_1$ and $s \ne s_2$ and $\alpha'(s_1) = \alpha(s_1) + \epsilon$ and $\alpha'(s_2) = \alpha(s_2) - \epsilon$. Then α and α' differ at exactly two points and $\|\alpha - \lambda\| = \|\alpha - \alpha'\| + \|\alpha' - \lambda\|$. By repeatedly applying the above, we obtain $\alpha_0, \ldots, \alpha_n$ in $P(S)$ such that $\alpha_0 = \alpha$, $\alpha_n = \lambda$, $\|\alpha - \lambda\| = \Sigma\, \|\alpha_{i-1} - \alpha_i\|$ and such that α_{i-1} and α_i differ at exactly two points.

If we choose $\nu_1, \ldots, \nu_n$ in $P(S \times T)$ such that $\pi\nu_i = (\alpha_i, \beta)$ and $\|\nu_{i-1} - \nu_i\| = \|\alpha_{i-1} - \alpha_i\|$ where $\nu_0 = \mu$, then ν_n satisfies $\pi\nu_n = (\lambda, \beta)$ and $\|\mu - \nu_n\| \le \|\alpha - \lambda\|$. Now apply the above process to ρ and β to obtain $\nu \in$

$P(S \times T)$ such that $\pi\nu = (\lambda, \rho)$ and $\|\nu_n - \nu\| \le \|\beta - \rho\|$. Then $\|\mu - \nu\| \le \|\mu - \nu_n\| + \|\nu_n - \nu\| \le \|\alpha - \lambda\| + \|\beta - \rho\|$.

The above combinatorial result extends to the case where S and T are Hausdorff spaces, since the set of finite convex combinations of point mass measures on S is weak* dense in $P(S)$.

Theorem 2.2. *Let $\mu \in P(S \times T)$ and set $\pi(\mu) = (\alpha, \beta)$. If $(\lambda, \rho) \in P(S) \times P(T)$, then there exists $\nu \in P(S \times T)$ satisfying $\pi(\nu) = (\lambda, \rho)$ and $\|\mu - \nu\| \le \|\alpha - \lambda\| + \|\beta - \rho\|$. Hence, π is a norm open mapping of $P(S \times T)$ onto $P(S) \times P(T)$.*

Proof. Let $\mathfrak{D}(S)$ denote the family of finite decompositions $\{U_1, \ldots, U_n\}$ of S where $U_1, \ldots, U_n$ are nonempty, disjoint Borel subsets of S. Given decompositions $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_m\}$ of S, we write $\{A_1, \ldots, A_n\} \le \{B_1, \ldots, B_m\}$ if each B_j is contained in some A_i. The relation $\le$ directs $\mathfrak{D}(S)$. Given $\mathfrak{U} = \{U_1, \ldots, U_n\}$ in $\mathfrak{D}(S)$ and $\mathfrak{V} = \{V_1, \ldots, V_m\}$ in $\mathfrak{D}(T)$, choose $x_i \in U_i$ and $y_i \in V_j$. Fix $\mu \in P(S \times T)$ and $(\lambda, \rho) \in P(S) \times P(T)$ and set $\pi(\mu) = (\alpha, \beta)$. Given $\mathfrak{U} = \{U_1, \ldots, U_n\}$ in $\mathfrak{D}(S)$ and $\mathfrak{V} = \{V_1, \ldots, V_m\}$ in $\mathfrak{D}(T)$ with choice points $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, define

$$\mu(\mathfrak{U}, \mathfrak{V}) = \sum_{i,j} \mu(U_i \times V_j) \cdot \delta(x_i, y_j)$$

where $\delta(x_i, y_j)$ is the point mass at (x_i, x_j). Thus, $(\mathfrak{U}, \mathfrak{V}) \to \mu(\mathfrak{U}, \mathfrak{V})$ is a net and $\mu(\mathfrak{U}, \mathfrak{V})$ converges to μ weak*. Likewise, define $\alpha(\mathfrak{U}) = \Sigma\alpha(U_i)\delta(x_i)$ and $\beta(\mathfrak{V}) = \Sigma\beta(V_j)\delta(y_j)$ and similarly for $\lambda(\mathfrak{U})$ and $\rho(\mathfrak{V})$.

We obtain ν as follows. By Lemma 5.1, there exists $\nu_{\mathfrak{U},\mathfrak{V}}$ such that $\pi(\nu_{\mathfrak{U},\mathfrak{V}}) = (\lambda(\mathfrak{U}), \rho(\mathfrak{V}))$ and

$$\begin{aligned}\|\mu(\mathfrak{U}, \mathfrak{V}) - \nu_{\mathfrak{U},\mathfrak{V}}\| &\le \|\alpha(\mathfrak{U}) - \lambda(\mathfrak{U})\| + \|\beta(\mathfrak{V}) - \rho(\mathfrak{V})\| \\ &\le \|\alpha - \lambda\| + \|\beta - \rho\|.\end{aligned}$$

The net $\nu_{\mathfrak{U},\mathfrak{V}}$ has a limit point ν since $P(S \times T)$ is weak* compact. Necessarily, we have $\nu_{\mathfrak{U},\mathfrak{V}}$ converges to ν weak*. Hence, by weak* continuity of π, we have $\pi(\nu) = (\lambda, \rho)$. Also, $\|\mu - \nu\| \le \|\alpha - \lambda\| + \|\beta - \rho\|$.

We next prove that π is weak* open. This requires a more delicate argument but again the proof rests upon Lemma 2.1.

Theorem 2.3. *Let S and T be compact Hausdorff spaces. Equip $P(S)$, $P(T)$ and $P(S \times T)$ with the weak* topology. The mapping π: $P(S \times T) \to P(S) \times P(T)$ is weak* open.*

Proof. Fix $\mu \in P(S \times T)$ and set $\pi(\mu) = (\alpha, \beta)$. Let $F_k \in C(S \times T)$ such that $0 \leq F_k \leq 1$ for $k = 1, \ldots, n$. Fix $1 > \epsilon > 0$. Set $\Omega = \{\nu \in P(S \times T) > |(\mu - \nu)F_k| < 18\epsilon$ for $k = 1, \ldots, n\}$. It suffices to show that $\pi(\Omega)$ is a weak* neighborhood of (α, β). There exist closed disjoint subsets $K_1, \ldots, K_p$ of S and $L_1, \ldots, L_q$ of T satisfying (1) $\omega(F_k, K_i \times L_j) < \epsilon/2$ where $\omega(F_k, K_i \times L_j)$ is the oscillation of F_k on $K_i \times L_j$ and (2) $\mu(K \times L) > 1 - \epsilon/2$ where $K = \bigcup K_i$ and $L = \bigcup L_j$. Now choose G_k continuous on $S \times T$ such that $0 \leq G_k \leq 1$, G_k is constant on a neighborhood of $K_i \times L_j$ for each i and j and $\|F_k - G_k\| < \epsilon/2$. Thus, $\{\nu\colon |(\mu - \nu)G_k| < 17\epsilon$ for each $k = 1, \ldots, n\}$ is contained in Ω. Choose disjoint open sets $U_1, \ldots, U_p$ and $V_1, \ldots, V_q$ such that $K_i \subseteq U_i$, $L_j \subseteq V_j$ and G_k is constant on $U_i \times V_j$ for each i, j and k. Set $U = \bigcup U_i$ and $V = \bigcup V_j$. Choose functions $f_1, \ldots, f_p$ in $C(S)$ and $g_1, \ldots, g_q$ in $C(T)$ satisfying $0 \leq f_i \leq 1$, $f_i = 1$ on K_i and $f_i = 0$ off U_i for $i = 1, \ldots, p$ and similarly for $g_1, \ldots, g_q$.

Now assume $(\lambda, \rho) \in P(S) \times P(T)$ and satisfies $\Sigma_i |(\alpha - \lambda)f_i| < \epsilon/2$ and $\Sigma_j |(\beta - \rho)g_j| < \epsilon/2$. The proof will be completed if we produce ν such that $\pi(\nu) = (\lambda, \rho)$ and $|(\mu - \nu)G_k| < 17\epsilon$ for $k = 1, \ldots, n$. We approximate μ, ρ, λ in norm and apply the above lemma. First, notice that $\alpha(K) - \lambda(U) \leq (\alpha - \lambda)(\Sigma f_i) < \epsilon/2$. Thus, $1 < \alpha(K) + \epsilon/2 < \lambda(U) + \epsilon$ and so $\lambda(U) > 1 - \epsilon$. Similarly, $\rho(V) > 1 - \epsilon$. Now define $\overline{\lambda}$ and $\overline{\rho}$ by $\lambda(U)\cdot\overline{\lambda} = \lambda|U$ and $\rho(V)\cdot\overline{\rho} = \rho|V$. Define $\overline{\mu}$ by $\mu(K \times L)\cdot\overline{\mu} = \mu|(K \times L)$. If θ is a probability measure and E is a θ-measurable set with $\theta(E) > 0$, then $\theta(E)\cdot\overline{\theta} = \theta|E$ implies $\|\theta - \overline{\theta}\| = 2[1 - \theta(E)]$. Hence, $\|\mu - \overline{\mu}\| < \epsilon$, $\|\lambda - \overline{\lambda}\| < 2\epsilon$ and $\|\rho - \overline{\rho}\| < 2\epsilon$. It suffices to find $\overline{\nu}$ satisfying $|(\overline{\mu} - \overline{\nu})G_k| < 14\epsilon$ for $k = 1, \ldots, n$ and $\pi(\overline{\nu}) = (\overline{\lambda}, \overline{\rho})$. To see that this is sufficient, set $m = \min\{\lambda(U), \rho(V)\}$. Then $\|\lambda - m\overline{\lambda}\| = 1 - m < \epsilon$. Similarly, $\|\rho - m\overline{\rho}\| < \epsilon$. Finally, set

$$\nu = m\overline{\nu} + (1 - m)^{-1} \cdot (\lambda - m\overline{\lambda}) \times (\rho - m\overline{\rho}).$$

We have $\pi(\nu) = (\lambda, \rho)$. Also,

$$|(\mu - \nu)G_k| \leq \|\mu - \overline{\mu}\| + |(\overline{\mu} - \overline{\nu})G_k| + \|\nu - \overline{\nu}\| < 17\epsilon.$$

We now construct $\overline{\nu}$. Set $\pi(\overline{\mu}) = (\overline{\alpha}, \overline{\beta})$. Then $\|\alpha - \overline{\alpha}\| < \epsilon$ and $\|\beta - \overline{\beta}\| < \epsilon$. Set $\alpha_i = \overline{\alpha}(K_i)$, $\beta_j = \overline{\beta}(L_j)$, $\lambda_i = \overline{\lambda}(U_i)$, $\rho_j = \overline{\rho}(V_j)$ and $\mu_{ij} = \overline{\mu}(K_i \times L_j)$. Then

$$\begin{aligned} \alpha_i - \lambda_i &= \overline{\alpha}(K_i) - \overline{\lambda}(U_i) \\ &= (\overline{\alpha} - \alpha)f_i + (\alpha - \lambda)f_i + (\lambda - \overline{\lambda})f_i + \overline{\lambda}(f_i) - \overline{\lambda}(U_i). \end{aligned}$$

Summing from $i = 1, \ldots, p$ and using $\Sigma\overline{\lambda}(U_i) = 1 > \overline{\lambda}(\Sigma f_i)$, we obtain

$$\sum|\alpha_i - \lambda_i| \leq |\bar{\alpha} - \alpha|\left(\sum f_i\right) + |\lambda - \bar{\lambda}|\left(\sum f_i\right)$$
$$+ \sum|(\alpha - \lambda)f_i| + 1 - \bar{\lambda}\left(\sum f_i\right)$$
$$< \epsilon + 2\epsilon + \epsilon/2 + 1 - \bar{\lambda}\left(\sum f_i\right).$$

We now estimate the last term. Namely, we have

$$1 - \bar{\lambda}\left(\sum f_i\right) = (\bar{\alpha} - \alpha)\left(\sum f_i\right) + (\alpha - \lambda)\left(\sum f_i\right) + (\lambda - \bar{\lambda})\left(\sum f_i\right)$$
$$< \epsilon + \epsilon/2 + 2\epsilon.$$

Thus, $\Sigma|\alpha_i - \lambda_i| < 7\epsilon$. Similarly, $\Sigma|\beta_j - \rho_j| < 7\epsilon$. Apply Lemma 5.1 to obtain a doubly stochastic matrix $[\nu_{ij}]$ such that $\Sigma_{i,j}|\mu_{ij} - \nu_{ij}| < 14\epsilon$, $\lambda_i = \Sigma_j \nu_{ij}$ and $\rho_j = \Sigma_i \nu_{ij}$. Define

$$\bar{\nu} = \sum_{i,j} \nu_{ij}(\bar{\lambda}|U_i) \times (\bar{\rho}|V_j)/\lambda_i\rho_j.$$

Then $\pi(\bar{\nu}) = (\bar{\lambda}, \bar{\rho})$. Also,

$$|(\bar{\mu} - \bar{\nu})G_k| = \left|\sum_{i,j} g^k_{ij}(\mu_{ij} - \nu_{ij})\right| < 14\epsilon$$

where g^k_{ij} is the constant value of G_k on $U_i \times V_j$. This completes the proof.

If one considers π on the cone of nonnegative measures on $S \times T$, then one obtains the following open mapping results.

Theorem 2.4. *Let* $\mu \in M^+(S \times T)$ *and set* $\pi\mu = (\alpha, \beta)$.

(1) *If* $(\lambda, \rho) \in M^+(S, T)$, *then there exists* $\nu \in M^+(S \times T)$ *satisfying* $\pi\nu = (\lambda, \rho)$ *and* $\|\mu - \nu\| \leq 3(\|\alpha - \lambda\| + \|\beta - \rho\|)/2$.

(2) *The mapping* $\pi\colon M^+(S \times T) \to M^+(S, T)$ *is weak* open.*

A proof for the first part may be obtained by establishing an appropriate variant of Lemma 2.1 [see Theorem 3.2(2) below], and taking limits in the weak* topology. The second part follows from Theorem 2.3.

3. Integrable functions on a probability space. Let (X, μ) be a probability space and let $P(X, \mu) = \{f \in L_1(X, \mu): f \geq 0 \text{ and } \|f\| = 1\}$ denote the nonnegative integrable functions on X of mass 1. We obtain open mapping results for the space $P(X, \mu)$ which are analogs of those in §2 and in [1]. We first consider a naturally induced operator of fundamental importance in probability theory. Let $(X, \mathfrak{A}, \mu)$ be a probability space and let $\mathfrak{B}$ be a σ-subalgebra of $\mathfrak{A}$. Let $\pi\colon L_1(X, \mathfrak{A}, \mu) \to L_1(X, \mathfrak{B}, \mu)$ be the conditional expectation operator for $\mathfrak{B}$. If σ denotes the inclusion mapping of $L_1(X, \mathfrak{B}, \mu)$ into $L_1(X, \mathfrak{A}, \mu)$, then π is the continuous extension of the adjoint σ^*: $L_\infty(X, \mathfrak{A}, \mu) \to L_\infty(X, \mathfrak{B}, \mu)$. See Moy [4].

Theorem 3.1. *Let $(X, \mathfrak{A}, \mu)$ be a probability space and let $\mathfrak{B}$ be a σ-subalgebra of $\mathfrak{A}$. Let π denote the conditional expectation operator for $\mathfrak{B}$.*

(1) *π preserves order intervals, i.e., if $f \in L_1^+(X, \mathfrak{A}, \mu)$ and if $\pi f \geq g \geq 0$ where $g \in L_1^+(X, \mathfrak{B}, \mu)$, then there exists $f \geq h \geq 0$ satisfying $\pi h = g$.*

(2) *Fix $f \in L_1^+(X, \mathfrak{A}, \mu)$ and $g \in L_1^+(X, \mathfrak{B}, \mu)$. There exists $h \in L_1^+(X, \mathfrak{A}, \mu)$ satisfying $\pi h = g$ and $\|f - h\| = \|\pi f - g\|$. Hence, $\pi: L_1^+(X, \mathfrak{A}, \mu) \to L_1^+(X, \mathfrak{B}, \mu)$ is norm open.*

(3) *$\pi: L_1^+(X, \mathfrak{A}, \mu) \to L_1^+(X, \mathfrak{B}, \mu)$ is open with respect to the weak topology.*

(4) *Let S and T denote the maximal ideal spaces of $L_\infty(X, \mathfrak{A}, \mu)$ and $L_\infty(X, \mathfrak{B}, \mu)$, respectively. There exists $\phi: S \to T$ such that ϕ is continuous and open and such that $(\sigma g)^\wedge = \hat{g} \circ \phi$ for each $g \in L_\infty(X, \mathfrak{B}, \mu)$ where $^\wedge$ is the Gelfand transform and σ is the inclusion mapping of $L_\infty(X, \mathfrak{A}, \mu)$.*

Proof. To verify (1), fix $f \in L_1^+(X, \mathfrak{A}, \mu)$ and assume $g \in L_1^+(X, \mathfrak{B}, \mu)$ where $\pi f \geq g$. Set $f_0 = 0$ and $g_0 = 0$. Now define f_n and g_n for $n = 1, 2, \ldots$ by the equations

$$f_{n+1} = \left(f - \sum_{i=0}^{n} f_i\right) \wedge \sigma\left(g - \sum_{i=0}^{n} g_i\right) \quad \text{and} \quad g_{n+1} = \pi f_{n+1}.$$

Now set $f_\infty = \Sigma_{i=1}^\infty f_i$ and $g_\infty = \Sigma_{i=1}^\infty g_i$. We have $f \geq f_\infty$ and $g \geq g_\infty$. Since $(f - f_\infty) \wedge \sigma(g - g_\infty) = 0$ and $\pi(f - f_\infty) \geq g - g_\infty$, we have $g = g_\infty$. Thus, π preserves order intervals. Also, notice that (1) $\Rightarrow$ (2).

Fix $f \in P(X, \mathfrak{A}, \mu)$ and let $\mathfrak{U}$ be a weak neighborhood of f in $P(X, \mathfrak{A}, \mu)$. To verify (3), we only need to show that $\pi\mathfrak{U}$ is a weak neighborhood of πf in $P(X, \mathfrak{B}, \mu)$. Choose disjoint sets $E_1, \ldots, E_n$ in $\mathfrak{A}$ which cover X and $\epsilon > 0$ satisfying $\{g \in P(X, \mathfrak{A}, \mu): |\langle g - f, \chi E_i\rangle| < \epsilon \text{ for } i = 1, \ldots, n\} \subseteq \mathfrak{U}$ and $\lambda_i = \int_{E_i} f\, d\mu > 0$ for $i = 1, \ldots, n$. We define $\langle h, k\rangle = \int h \cdot k\, d\mu$ for $h \in L_1$ and $k \in L_\infty$. Set $f_i = \lambda_i^{-1} f \cdot \chi E_i$. Then $\Sigma_{i=1}^n \lambda_i f_i = f$ and $\Sigma_{i=1}^n \lambda_i = 1$ and $f_i \in P(X, \mathfrak{A}, \mu)$. Now set $\mathfrak{U}_j = \{g \in P(X, \mathfrak{A}, \mu): |\langle g - f_j, \chi E_i\rangle| < \epsilon \text{ for } i = 1, \ldots, n\}$. Then $\lambda_1\mathfrak{U}_1 + \cdots + \lambda_n\mathfrak{U}_n \subseteq \mathfrak{U}$. Since $f_j \cdot \chi E_i = 0$ if $i \neq j$, we have that $\mathfrak{U}_j = \{g \in P(X, \mathfrak{A}, \mu): |\langle g, \chi E_j\rangle| > 1 - \epsilon\}$ and so, by applying (1), that $\pi\mathfrak{U}_j$ is a weak neighborhood of πf_j in $P(X, \mathfrak{B}, \mu)$. It remains to show that if $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are weak neighborhoods of g_1 and g_2 in $P(X, \mathfrak{B}, \mu)$ and if $\alpha_1 + \alpha_2 = 1$ where $\alpha_1, \alpha_2 > 0$, then $\alpha_1\mathfrak{V}_1$ and $\alpha_2\mathfrak{V}_2$ is a weak neighborhood of $\alpha_1 g_1 + \alpha_2 g_2$. Fix $\psi_1, \ldots, \psi_m \in L_\infty(X, \mathfrak{B}, \mu)$ such that $\|\psi_j\|_\infty \leq 1$ and $\mathfrak{V}_i \supseteq \{g \in P(X, \mathfrak{B}, \mu): |\langle g - g_i, \psi_j\rangle| < \epsilon \text{ for } j = 1, \ldots, m\}$. Set $\phi_i = g_i/(\alpha_1 g_1 + \alpha_2 g_2)$. Set $\alpha = \max\{\alpha_1/\alpha_2, \alpha_2/\alpha_1\}$. Fix $g \in P(X, \mathfrak{B}, \mu)$ satisfying

$$|\langle g - (\alpha_1 g_1 + \alpha_2 g_2), \phi_i \psi_j \rangle| < \epsilon/2 \quad \text{and} \quad \|\phi_i g\| < 1 + \epsilon/2\alpha$$

for $i = 1, 2$ and $j = 1, \ldots, m$. We now have $\langle \phi_i g - g_i, \psi_j \rangle = \langle g - (\alpha_1 g_1 + \alpha_2 g_2), \phi_i \psi_j \rangle$ and $\alpha_1 \phi_1 g + \alpha_2 \phi_2 g = g$. Assume $\beta = \|\phi_1 g\| \geq 1$. Then set $h_1 = \phi_1 g/\beta$ and $h_2 = \phi_2 = \phi_2 g + \alpha_1(\phi_1 g - h_1)/\alpha_2$. Then $\alpha_1 h_1 + \alpha_2 h_2 = g$ and $|\langle h_i - g_i, \psi_j \rangle| < \epsilon$ since $\|\alpha_1(\phi_1 g - h_1)/\alpha_2\| < \epsilon/2$ and $\|\phi_1 g_1 - h_1\| < \epsilon/2$. The case $\|\phi_1 g\| < 1$ is similar.

To prove (4), let $S^{\#}$ and $T^{\#}$ denote the open and closed subsets of S and T, respectively. Define $\hat{\pi}$: $C(S) \to C(T)$ and $\hat{\sigma}$: $C(T) \to C(S)$ by $\hat{\pi}(\hat{f}) = (\pi f)\hat{}$ for $f \in L_\infty(X, \mathfrak{A}, \mu)$ and $\hat{\sigma}(\hat{g}) = (\sigma g)\hat{}$ for $g \in L_\infty(X, \mathfrak{B}, \mu)$. Thus, $\hat{\sigma}$ maps idempotents in $C(T)$ to idempotents in $C(S)$. Thus, we have Φ: $T^{\#} \to S^{\#}$ defined by $\Phi(E) = K$ iff $\hat{\sigma}(\chi_E) = \chi_K$. Since Φ is a Boolean algebra monomorphism of T into S, Φ determines [2, p. 85] a continuous map ϕ of S onto T where $\phi^{-1}(E) = \Phi(E)$ for each $E \in T^{\#}$. Thus, $g \circ \phi = \hat{\sigma}(g)$ for each $g \in C(T)$. To see that ϕ is open, fix K open and closed in S. Let $U = \{t \in T: (\pi\chi_K)(t) > 0\}$. Then U is open in T and $U \subseteq \phi(K) \subseteq \overline{U}$ where $\overline{U}$ denotes the closure of U. Since T is extremally disconnected, we have $\overline{U}$ is open and so $\phi(K) = \overline{U}$ is open.

We now take up the problem of establishing open mapping properties for marginals of integrable functions on $(X \times Y, \mu \times \nu)$ where (X, μ) and (Y, ν) are probability spaces. Given $F \in L_1(X \times Y, \mu \times \nu)$, set $\pi F = (f_1, f_2)$ where $f_1(x) = \int F(x, y)\, d\nu(y)$ and $f_2(y) = \int F(x, y)\, d\mu(x)$. Although closely related to Theorem 2.2, our next result requires a different proof. A straightforward application of Lemma 2.1 shows that π: $P(X \times Y, \mu \times \nu) \to P(X, \mu) \times P(Y, \nu)$ is open with respect to the weak topology. Likewise, in case S and T are totally disconnected compact spaces, one can similarly show that π: $P(S \times T) \to P(S) \times P(T)$ is weak* open. One should also note that part (1) implies Lemma 2.1.

Theorem 3.2. *Let (X, μ) and (Y, ν) be probability spaces. Let $F \in L_1^+(X \times Y, \mu \times \nu)$ and set $\pi F = (f_1, f_2)$.*

(1) *Assume $\|F\| = 1$ and $(g_1, g_2) \in P(X, \mu) \times P(Y, \nu)$. Then there exists $G \geq 0$ such that $\pi G = (g_1, g_2)$ and $\|F - G\| \leq \|f_1 - g_1\| + \|f_2 - g_2\|$.*

(2) *Fix nonnegative marginals (g_1, g_2). There exists $G \geq 0$ such that $\pi G = (g_1, g_2)$ and $\|F - G\| \leq 3(\|f_1 - g_1\| + \|f_2 - g_2\|)/2$.*

(3) *The mapping π: $P(X \times Y, \mu \times \nu) \to P(X, \mu) \times P(Y, \nu)$ is open with respect to the weak topology.*

Proof. As noted above, the proof of (3) is straightforward. Fix $F \geq 0$ and set $\pi F = (f_1, f_2)$. Let (g_1, g_2) be nonnegative marginals. We first consider the case where $f_2 = g_2$. By Theorem 3.1(1), there exists $0 \le H \le F$ such that $\pi H = (h_1, h_2)$ where $h_1 = f_1 \wedge g_1$. We may certainly assume $\|f_1 - h_1\| = \lambda > 0$. Set $G = H + (g_1 - h_1)(g_2 - h_2)/\lambda$. One obtains $\pi G = (g_1, g_2)$ and $\|F - G\| = 2\|g_1 - h_1\| = \|f_1 - g_1\|$. The case $f_1 = g_1$ is handled similarly. To see (1), simply consider the intermediate marginals (f_1, g_2) and apply the above two cases.

We next consider the case $(g_1, g_2) \le (f_1, f_2)$ which denotes $g_1 \le f_1$ and $g_2 \le f_2$. Choose $F \ge B \ge 0$ such that B is maximal with respect to property $\pi B \le (g_1, g_2)$. Set $\pi B = (b_1, b_2)$ and $h_i = g_i - b_i$. We may assume $\|h_i\| > 0$. Next, choose $F - B \ge C \ge 0$ such that C is maximal with respect to the property $\pi(F - B - C) \ge (h_1, h_2)$. If we set $A = F - B - C$ and $\pi A = (a_1, a_2)$ and $\pi C = (c_1, c_2)$, then we have $A = 0$ on $[h_1(x) > 0] \times [h_2(y) > 0]$ and on $[h_1(x) = 0] \times [h_2(y) = 0]$ and we have $h_i = a_i$ on $[h_i > 0]$. Thus,

$$\|f_1 - g_1\| + \|f_2 - g_2\| = \|c_1\| + \|c_2\| + \|a_1 - h_1\| + \|a_2 - h_2\|.$$

Then $\pi H = (h_1, h_2)$. It suffices to show

$$\|A - H\| \le \frac{3}{2}(\|a_1 - h_1\| + \|a_2 - h_2\|)$$

since setting $G = B + H$ yields $\pi G = (g_1, g_2)$ and

$$\|F - G\| \le \|C\| + \|A - H\| \le \frac{3}{2}(\|f_1 - g_1\| + \|f_2 - g_2\|).$$

Using $\|h_1\| = \|a_2 - h_2\|$ and $\|h_2\| = \|a_1 - h_1\|$, we have

$$\|A - H\| = \|A\| + \|H\| = \|a_1 - h_1\| + \|h_2\| = \frac{3}{2}(\|a_1 - h_1\| + \|a_2 - h_2\|).$$

We now complete the proof of (2). Set $k_i = f_i \wedge g_i$. Set $k_i = f_i \wedge g_i$. We may assume $\|k_1\| \le \|k_2\|$. Fix $1 \ge \epsilon, \delta \ge 0$ such that $(k_1 + \epsilon(f_1 - k_1), k_2)$, and $(k_1 + \delta(g_1 - k_1), k_2)$ are marginals. By the case considered above, there exists $K \ge 0$ satisfying $\pi K = (k_1 + \epsilon(f_1 - k_1), k_2)$ and

$$\|F - K\| \le \frac{3}{2}(\|f_2 - k_2\| + (1 - \epsilon)\,\|f_1 - k_1\|).$$

Now apply (1) to obtain H satisfying $\pi H = (k_1 + \delta(g_1 - k_1), k_2)$ and $\|K - H\| \le \epsilon\|f_1 - k_1\| + \delta\|g_1 - k_1\|$. Choose $G \ge 0$ satisfying $\pi G = (g_1, g_2)$ and $\|H - G\| = (1 - \delta)\|g_1 - k_1\|$. Then we have

$$\|F - G\| \le \|F - K\| + \|K - H\| + \|H - G\| \le \frac{3}{2}(\|f_1 - g_1\| + \|f_2 - g_2\|).$$

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REFERENCES

1. S. Z. Ditor and L. Q. Eifler, *Some open mapping theorems for measures*, Trans. Amer. Math. Soc. 164 (1972), 287–293.

2. P. R. Halmos, *Lectures on Boolean algebras*, Van Nostrand Math. Studies, no. 1, Van Nostrand, Princeton, N. J., 1963. MR 29 #4713.

3. H. G. Kellerer, *Masstheoretische Marginalprobleme*, Math. Ann. 153 (1964), 168–198. MR 28 #5160.

4. S.-T. C. Moy, *Characterizations of conditional expectation as a transformation on function spaces*, Pacific J. Math. 4 (1954), 47–63. MR 15, 722.

5. V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Statist. 36 (1965), 423–439. MR 31 #1693.

6. L. Q. Eifler, *Open mapping theorems for probability measures on metric spaces* (to be submitted to Ann. Inst. Fourier, Grenoble).

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